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COMMENTS ON "VIBRATION ANALYSIS OF ARBITRARY SHAPED MEMBRANES USING NON-DIMENSIONAL DYNAMIC INFLUENCE FUNCTION"

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In reference [1], the authors presented the so-called non-dimensional dynamic influence function method for membrane vibration. A regular formulation and singularity-free method were obtained. Also, a symmetry and meshless formulation can be achieved. The auxiliary system is a complementary solution instead of fundamental function. It is source free in the influence function. Many successful examples of the Dirichlet types were demonstrated. It seems that this method is very attractive. However, this method can be treated as one kind of the Trefftz method [2–4]. Based on the dual formulation developed by Chen and Hong [5, 6], the influence function is nothing but the imaginary part of the fundamental solution ($U(s, x) = iH_0^{(1)}(kr)$) [7]. The method by Kang *et al.* [1] can be treated as a special case of the imaginary-part dual BEM. Also, the real-part dual BEM was developed and many references can be referred [8, 9]. MRM formulation can also be viewed as a real-part formulation and its occurrence of spurious eigenvalues have been found in references [10–15]. It is well known that the real-part, imaginary-part formulations and multiple reciprocity method all result in spurious eigensolutions. Particularly, the imaginary-part formulation also results in an ill-posed problem since the condition number for the influence matrix is very large. Many approaches have been employed to filter out the spurious solutions and extract the true solution, for example, residue method [11, 12], singular-value decomposition (SVD) technique [9, 13–15], generalized singular-value decomposition (GSVD) technique [16] and domain partition technique [17]. It is expected that the Kang's method has the problems of spurious solutions and ill-conditioned behavior since it is an imaginary-part formulation. However, no such information was addressed. Some points will be discussed as follows.

(1) *Spurious eigensolution*: It is interesting to find that all the examples in reference [1] are of the Dirichlet type. According to the theoretical derivation, the Neumann problem has the problems of spurious solutions using the imaginary-part formulation. We will prove that in the following.

As mentioned earlier, spurious eigenvalues occur in the real-part of MRM formulation [9, 11, 12]. Also, the imaginary-part BEM results in spurious solutions [7]. Here, we will derive the true and spurious solutions in the discrete system analytically for a circular domain by using the non-dimensional influence function method [1] and the imaginary-part dual BEM [7] in a unified way. The degenerate kernels and circulants are employed to study the discrete system in an exact form. The relation between the non-dimensional influence function method and the imaginary-part dual BEM is also addressed and is summarized in Table 1. The symbols in Table 1 follow the dual model of Chen and Hong [6].

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TABLE 1

Comparisons of non-dimensional influence function method and the imaginary-part dual BEM

Method	Imaginary-part dual BEM by Chen *et al.* [6]	Non-dimensional dynamic influence function by Kang *et al.* [1]
Auxiliary system	$\mathrm{J}_0(k\lvert \mathrm{x}-\mathrm{s}\rvert)$	$\mathrm{J}_0(k\lvert \mathrm{x}-\mathrm{s}\rvert)$
Density	Distributed on boundary using constant element	Concentrated on discrete points
Solution representation for field or boundary data	$0=\sum_{j=1}^{N}\int_{B_j}\{(T^I(s_j,x_i)u(s_j) - U^I(s_j,x_i)t(s_j)\}\,\mathrm{d}B(s_j)$ $0=\sum_{j=1}^{N}\int_{B_j}\{(M^I(s_j,x_i)u(s_j) - L^I(s_j,x_i)t(s_j)\}\,\mathrm{d}B(s_j)$	$u(x_i)=\sum \mathrm{J}_0(k\lvert x_i-s_j\rvert)A_j$ $u(x_i)=\sum \frac{\partial \mathrm{J}_0(k\lvert x_i-s_j\rvert)}{\partial n_{s_j}}B_j$
Eigenequation for Dirichlet problem	UT method: $U^I_{ij}t_j=0$ LM method: $L^I_{ij}t_j=0$	UL method: $(SM)_{ij}A_j=0$ TM method: $(SM_s)_{ij}B_j=0$
Eigenequation for Neumann problem	UT method: $T^I_{ij}u_j=0$ LM method: $M^I_{ij}u_j=0$	UL method: $(SM_x)_{ij}A_j=0$ TM method: $(SM_{xs})_{ij}B_j=0$
Influence matrix	$U^I_{ij}=\int_{B_j}U(s_j,x_i)\,\mathrm{d}B(s_j)$ $T^I_{ij}=\int_{B_j}T(s_j,x_i)\,\mathrm{d}B(s_j)$ $L^I_{ij}=\int_{B_j}L(s_j,x_i)\,\mathrm{d}B(s_j)$ $M^I_{ij}=\int_{B_j}M(s_j,x_i)\,\mathrm{d}B(s_j)$	$(SM)_{ij}=\mathrm{J}_0(k\lvert x_i-s_j\rvert)$ $(SM_s)_{ij}=\frac{\partial \mathrm{J}_0(k\lvert x_i-s_j\rvert)}{\partial n_{s_j}}$ $(SM_x)_{ij}=\frac{\partial \mathrm{J}_0(k\lvert x_i-s_j\rvert)}{\partial n_{x_i}}$ $(SM_{sx})_{ij}=\frac{\partial^2 \mathrm{J}_0(k\lvert x_i-s_j\rvert)}{\partial n_{x_i}\partial n_{s_j}}$
Spurious eigenequation for Dirichlet problem	*UT method*: $\mathrm{J}_n(k\rho)=0$ *LM method*: $\mathrm{J}'_n(k\rho)=0$	*UL method*: $\mathrm{J}_n(k\rho)=0$* *TM method*: $\mathrm{J}'_n(k\rho)=0$
True eigenequation for Dirichlet problem	*UT method*: $\mathrm{J}_n(k\rho)=0$ *LM method*: $\mathrm{J}_n(k\rho)=0$	*UL method*: $\mathrm{J}_n(k\rho)=0$* *TM method*: $\mathrm{J}_n(k\rho)=0$
Spurious eigenequation for Neumann problem	*UT method*: $\mathrm{J}_n(k\rho)=0$ *LM method*: $\mathrm{J}'_n(k\rho)=0$	*UL method*: $\mathrm{J}_n(k\rho)=0^\dagger$ *TM method*: $\mathrm{J}'_n(k\rho)=0$
True eigenequation for Neumann problem	*UT method*: $\mathrm{J}'_n(k\rho)=0$ *LM method*: $\mathrm{J}'_n(k\rho)=0$	*UL method*: $\mathrm{J}'_n(k\rho)=0^\dagger$ *TM method*: $\mathrm{J}'_n(k\rho)=0$
Condition number	$\frac{\mathrm{Max}(h_n)}{\mathrm{Min}(h_n)},\ n=0,1,2,\ldots$	$\frac{\mathrm{Max}(h_n)}{\mathrm{Min}(h_n)},\ n=0,1,2,\ldots$

*Example is available in reference [1].
†Example is not available in reference [1].

If the imaginary-part dual BEM is employed, the dual boundary integral equations are obtained as follows [7]:

$$0 = \int_B T^I(s, x)u(s)\,\mathrm{d}B(s) - \int_B U^I(s, x)t(s)\,\mathrm{d}B(s), \tag{1}$$

$$0 = \int_B M^I(s, x)u(s)\,\mathrm{d}B(s) - \int_B L^I(s, x)t(s)\,\mathrm{d}B(s), \tag{2}$$

where u and t are the potential and its normal derivative, the four imaginary-part kernels in the dual formulation can be expressed in terms of degenerate kernels [7, 18, 19] as shown below:

$$U^I(s, x) = \frac{-\pi}{2} J_0(kr) = U(\theta, \phi) = -\sum_{m=-\infty}^{\infty} \frac{\pi}{2} \mathrm{J}_m(k\rho)\mathrm{J}_m(k\rho)\cos(m(\theta - \phi)), \tag{3}$$

$$T^I(s, x) = T(\theta, \phi) = -\sum_{m=-\infty}^{\infty} \frac{\pi k}{2} \mathrm{J}'_m(k\rho)\mathrm{J}_m(k\rho)\cos(m(\theta - \phi)), \tag{4}$$

$$L^I(s, x) = T(\theta, \phi) = -\sum_{m=-\infty}^{\infty} \frac{\pi k}{2} \mathrm{J}_m(k\rho)\mathrm{J}'_m(k\rho)\cos(m(\theta - \phi)), \tag{5}$$

$$M^I(s, x) = M(\theta, \phi) = -\sum_{m=-\infty}^{\infty} \frac{\pi k^2}{2} \mathrm{J}'_m(k\rho)\mathrm{J}'_m(k\rho)\cos(m(\theta - \phi)), \tag{6}$$

in which J is the Bessel function of the first kind, $x = (\rho, \phi)$ and $s = (\rho, \theta)$.

For the non-dimensional influence function method, the representation for the solution can be expressed as

$$\text{UL method} \quad u(x_i) = \sum_j \mathrm{J}_0(k|x_i - s_j|)A_j = (SM)_{ij}A_j, \tag{7}$$

$$\text{TM method} \quad u(x_i) = \sum_j \frac{\partial \mathrm{J}_0(k|x_i - s_j|)}{\partial n_s} B_j = (SM_s)_{ij}B_j, \tag{8}$$

where A_j and B_j are the generalized unknowns using the UL and TM methods respectively. For simplicity, we consider the same problem of a circular domain in reference [1]. By superimposing $2N$ constant source distribution, u or t (or concentrated strength, A_j or B_j) along the real boundary with radius ρ and collocating the $2N$ points on the boundary with radius ρ, we have

$$[U^I] = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{2N-2} & a_{2N-1} \\ a_{2N-1} & a_0 & a_1 & \cdots & a_{2N-3} & a_{2N-2} \\ a_{2N-2} & a_{2N-1} & a_0 & \cdots & a_{2N-4} & a_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{2N-1} & a_0 \end{bmatrix}, \tag{9}$$

$$[T^I] = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{2N-2} & b_{2N-1} \\ b_{2N-1} & b_0 & b_1 & \cdots & b_{2N-3} & b_{2N-2} \\ b_{2N-2} & b_{2N-1} & b_0 & \cdots & b_{2N-4} & b_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_{2N-1} & b_0 \end{bmatrix}, \tag{10}$$

$$[L^I] = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{2N-2} & c_{2N-1} \\ c_{2N-1} & c_0 & c_1 & \cdots & c_{2N-3} & c_{2N-2} \\ c_{2N-2} & c_{2N-1} & c_0 & \cdots & c_{2N-4} & c_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_{2N-1} & c_0 \end{bmatrix}, \tag{11}$$

$$[M^I] = \begin{bmatrix} d_0 & d_1 & d_2 & \cdots & d_{2N-2} & d_{2N-1} \\ d_{2N-1} & d_0 & d_1 & \cdots & d_{2N-3} & d_{2N-2} \\ d_{2N-2} & d_{2N-1} & d_0 & \cdots & d_{2N-4} & d_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_1 & d_2 & d_3 & \cdots & d_{2N-1} & d_0 \end{bmatrix}, \tag{12}$$

where $[U^I]$, $\{T^I]$, $\{L^I]$, and $[M^I]$ are the influence matrices with the elements shown below:

$$a_m = \int_{(m-1/2)\Delta\theta}^{(m+1/2)\Delta\theta} U(\theta, 0)\rho\, \mathrm{d}\theta \approx U(\theta_m, 0)S, \quad m = 0, 1, 2, \ldots, 2N-1, \tag{13}$$

$$b_m = \int_{(m-1/2)\Delta\theta}^{(m+1/2)\Delta\theta} T(\theta, 0)\rho\, \mathrm{d}\theta \approx T(\theta_m, 0)S, \quad m = 0, 1, 2, \ldots, 2N-1, \tag{14}$$

$$c_m = \int_{(m-1/2)\Delta\theta}^{(m+1/2)\Delta\theta} L(\theta, 0)\rho\, \mathrm{d}\theta \approx L(\theta_m, 0)S, \quad m = 0, 1, 2, \ldots, 2N-1, \tag{15}$$

$$d_m = \int_{(m-1/2)\Delta\theta}^{(m+1/2)\Delta\theta} M(\theta, 0)\rho\, \mathrm{d}\theta \approx M(\theta_m, 0)S, \quad m = 0, 1, 2, \ldots, 2N-1, \tag{16}$$

in which $\Delta\theta = 2\pi/2N$, $\theta_m = m\Delta\theta$ and

$$S = \begin{cases} \rho\Delta\theta & \text{for the imaginary-part dual BEM by Chen } et\ al.\ [7], \\ 1 & \text{for the non-dimensional influence function method by Kang } et\ al.\ [1]. \end{cases} \tag{17}$$

For the non-dimensional influence function method, S is reduced to one since distribution is lumped on the concentrated point. The matrices, $[U^I]$, $\{T^I]$, $\{L^I]$ and $[M^I]$, are found to be in circulant forms since rotation symmetry for the influence coefficients exists. By introducing the following bases for the circulants [20]: I, C_{2N}^1, C_{2N}^2, $\ldots$, C_{2N}^{2N-1}, we can expand the four matrices into

$$[U^I] = a_0 I + a_1 C_{2N}^1 + a_2 C_{2N}^2 + \cdots + a_{2N-1} C_{2N}^{2N-1}, \tag{18}$$

$$[T^I] = b_0 I + b_1 C_{2N}^1 + b_2 C_{2N}^2 + \cdots + b_{2N-1} C_{2N}^{2N-1}, \tag{19}$$

$$[L^I] = c_0 I + c_1 C_{2N}^1 + c_2 C_{2N}^2 + \cdots + c_{2N-1} C_{2N}^{2N-1}, \tag{20}$$

$$[M^I] = d_0 I + d_1 C_{2N}^1 + d_2 C_{2N}^2 + \cdots + d_{2N-1} C_{2N}^{2N-1}, \tag{21}$$

where the superscript "I" denotes imaginary part and $[I]$ is a unit matrix and

$$C_{2N} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{2N \times 2N}. \tag{22}$$

Based on the similar properties of the matrices of $[U^I]$, $\{T^I]$, $\{L^I]$, $\{M^I]$ and $[C_{2N}]$ the eigenvalues can be derived as shown below:

$$\lambda_\ell = a_0 + a_1 \alpha_\ell + a_2 \alpha_\ell^2 + \cdots + a_{2N-1} \alpha_\ell^{2N-1}, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N, \tag{23}$$

$$\mu_\ell = b_0 + b_1 \alpha_\ell + b_2 \alpha_\ell^2 + \cdots + b_{2N-1} \alpha_\ell^{2N-1}, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N, \tag{24}$$

$$v_\ell = c_0 + c_1 \alpha_\ell + c_2 \alpha_\ell^2 + \cdots + c_{2N-1} \alpha_\ell^{2N-1}, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N, \tag{25}$$

$$\delta_\ell = d_0 + d_1 \alpha_\ell + d_2 \alpha_\ell^2 + \cdots + d_{2N-1} \alpha_\ell^{2N-1}, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N, \tag{26}$$

where λ_ℓ, μ_ℓ, v_ℓ and δ_ℓ are the eigenvalues for $[U^I]$, $\{T^I]$, $\{L^I]$ and $[M^I]$ respectively, and α_ℓ are the eigenvalues for the matrix $[C_{2N}]$. It is easily found that the eigenvalues, α_n, and eigenvectors, $\{\phi\}_n$, for the circulant $[C_{2N}]$ are the roots for $\alpha^{2N} = 1$ as shown below:

$$\alpha_n = e^{i(2\pi n/2N)}, \quad n = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N \text{ or } n = 0, 1, 2, \ldots, 2N-1, \tag{27}$$

$$\{\phi\}_n = \begin{Bmatrix} 1 \\ \alpha_n \\ \alpha_n^2 \\ \alpha_n^3 \\ \vdots \\ \alpha_n^{2N-1} \end{Bmatrix}, \quad n = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N \text{ or } n = 0, 1, 2, \ldots, 2N-1, \tag{28}$$

respectively.

Substituting equation (27) into equation (23), we have

$$\lambda_\ell = \sum_{m=0}^{2N-1} a_m \alpha_\ell^m = \sum_{m=0}^{2N-1} a_m e^{i(2\pi/2N)m\ell}, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N. \tag{29}$$

According to the definition for a_m in equations (3) and (13) we have

$$a_m = a_{2N-m}, \quad m = 0, 1, 2, \ldots, 2N-1. \tag{30}$$

Substituting equation (30) into equation (29), we have

$$\lambda_\ell = a_0 + (-1)^\ell a_N + \sum_{m=1}^{N-1} (\alpha_\ell^m + \alpha_\ell^{2N-m}) a_m$$

$$= \sum_{m=0}^{2N-1} \cos(m\ell\Delta\theta) a_m. \tag{31}$$

Substituting a_m in equation (13) into equation (31), we can transform the Riemann sum to the following integral:

$$\lambda_\ell \approx \sum_{m=0}^{2N-1} \cos(m\ell\Delta\theta) U(m\Delta\theta, 0) S = \int_0^{2\pi} \cos(\ell\theta) U(\theta, 0) \rho \, \mathrm{d}\theta \frac{S}{\rho\Delta\theta} \tag{32}$$

as N approaches infinity. By using the degenerate kernel for $U^I(s, x)$ in equation (3), equation (32) reduces to

$$\lambda_\ell = \frac{S}{\rho\Delta\theta} \int_0^{2\pi} \cos(\ell\theta) \sum_{m=-\infty}^{\infty} \frac{-\pi}{2} \mathrm{J}_m(k\rho)\mathrm{J}_m(k\rho) \cos m\theta \, \rho \, \mathrm{d}\theta$$

$$= -\frac{S}{\rho\Delta\theta} \pi^2 \rho \mathrm{J}_\ell(k\rho) J_\ell(k\rho). \tag{33}$$

Similarly, we have

$$\mu_\ell = -\frac{S}{\rho\Delta\theta} \pi^2 k\rho \mathrm{J}'_\ell(k\rho)\mathrm{J}_\ell(k\rho), \tag{34}$$

$$v_\ell = -\frac{S}{\rho\Delta\theta} \pi^2 k\rho \mathrm{J}_\ell(k\rho)\mathrm{J}'_\ell(k\rho), \tag{35}$$

$$\delta_\ell = -\frac{S}{\rho\Delta\theta} \pi^2 k^2 \rho \mathrm{J}'_\ell(k\rho)\mathrm{J}'_\ell(k\rho), \tag{36}$$

where μ_ℓ, v_ℓ and δ_ℓ are the eigenvalues of $[T^I]$, $[L^I]$ and $[M^I]$ matrices respectively. Since the wave number k is imbedded in each element of the circulant matrices, the corresponding eigenvalues for the four matrices are also functions of k. Finding the eigenvalues for the Helmholtz eigenproblem or finding the zeros for the determinant of the circulants is equal to finding the zeros for the multiplication of all their eigenvalues. The determinant can be obtained as follows:

$$det[U^I] = \lambda_0 \lambda_N (\lambda_1 \lambda_2 \cdots \lambda_{N-1})(\lambda_{-1}\lambda_{-2} \cdots \lambda_{-(N-1)}), \tag{37}$$

$$det[T^I] = \mu_0 \mu_N (\mu_1 \mu_2 \cdots \mu_{N-1})(\mu_{-1}\mu_{-2} \cdots \mu_{-(N-1)}), \tag{38}$$

$$det[L^I] = v_0 v_N (v_1 v_2 \cdots v_{N-1})(v_{-1} v_{-2} \cdots v_{-(N-1)}), \tag{39}$$

$$det[M^I] = \delta_0 \delta_N (\delta_1 \delta_2 \cdots \delta_{N-1})(\delta_{-1} \delta_{-2} \cdots \delta_{-(N-1)}). \tag{40}$$

Since the alternating properties for the Bessel function can be obtained, i.e.,

$$\mathrm{J}_{-\ell}(k\rho) = (-1)^{\ell} \mathrm{J}_{\ell}(k\rho), \quad \ell \in \mathrm{N}, \tag{41}$$

$$\mathrm{J}'_{-\ell}(k\rho) = (-1)^{\ell} \mathrm{J}'_{\ell}(k\rho), \quad \ell \in \mathrm{N}. \tag{42}$$

Equation (37)–(40) can be reduced to

$$det[U^I] = \lambda_0 (\lambda_1 \lambda_2 \cdots \lambda_{N-1})^2 \lambda_N, \tag{43}$$

$$det[T^I] = \mu_0 (\mu_1 \mu_2 \cdots \mu_{N-1})^2 \mu_N, \tag{44}$$

$$det[L^I] = v_0 (v_1 v_2 \cdots v_{N-1})^2 v_N, \tag{45}$$

$$det[M^I] = \delta_0 (\delta_1 \delta_2 \cdots \delta_{N-1})^2 \delta_N. \tag{46}$$

The squared terms in equations (43)–(46) imply that double roots occur for λ_ℓ when $\ell = 1, 2, \ldots, N-1$. In order to verify that either $\mathrm{J}_\ell(k\rho) = 0$ or $\mathrm{J}'_\ell(k\rho) = 0$ is a true eigenequation, the dual formulation (UT + LM or UL + TM) is needed to distinguish the true and spurious solutions.

The possible (true or spurious) eigenvalues occur at

$$\mathrm{J}_\ell(k\rho)\mathrm{J}_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N \tag{47}$$

for the Dirichlet problem using the UT method [7] or the UL method [1] since the determinant of $[U^I]$ matrix is zero using equations (33) and (37).

For the LM method [7] or the TM method [1], the possible (true or spurious) eigenvalues occur at

$$\mathrm{J}_\ell(k\rho)\mathrm{J}'_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N \tag{48}$$

for the Dirichlet problem since the determinant of $[L]$ matrix is zero using equations (35) and (39).

After comparing the results from the dual formulation in equations (47) and (48), we can determine the true and spurious eigenequation for the Dirichlet problem as follows:

$$\text{True eigenequation:} \quad \mathrm{J}_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N, \tag{49}$$

$$\text{Spurious eigenequation:} \quad \mathrm{J}'_\ell(k\rho) = 0, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N. \tag{50}$$

Equation (47) indicates that the Kang's method fortunately results in a spurious solution which is the same as the true solution for the Dirichlet problem. The reason why Kang *et al.* did not find the spurious solution is that only the Dirichlet cases were considered. However, it is not the case for the Neumann problem. Similarly, we can extend the Dirichlet problem to the Neumann problem for demonstrating that the spurious solutions occur in the Kang's method.

After comparing the results obtained by the dual formulation, we can summarize the spurious eigenequations for both the Dirichlet and Neumann problems:

$$\mathrm{J}_\ell(k\rho) = 0 \quad \text{using the } UT \text{ or the } UL \text{ equation,} \tag{51}$$

$$\mathrm{J}'_\ell(k\rho) = 0 \quad \text{using the } LM \text{ or the } TM \text{ equation.} \tag{52}$$

The true eigenequations using the $UT(UL)$ or the $LM(TM)$ method are found to be

$$\mathrm{J}_\ell(k\rho) = 0 \quad \text{for the Dirichlet problem,} \tag{53}$$

$$\mathrm{J}'_\ell(k\rho) = 0 \quad \text{for the Neumann problem.} \tag{54}$$

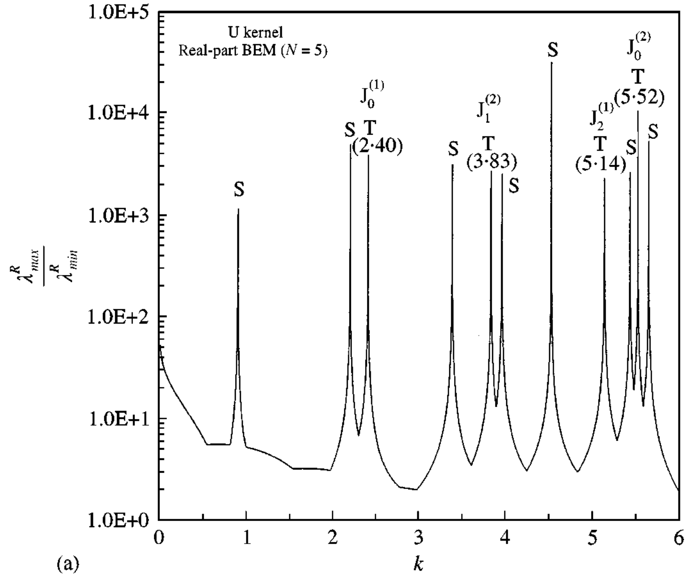
To demonstrate the above points, Figure 1(a) shows the condition number versus k using the real-part BEM ($N = 5$) for the Dirichlet problem. A spurious eigenvalue can be found from the position where the local maximum of condition number occurs. For the imaginary-part BEM, no spurious eigenvalues occur as shown in Figure 2(a) using $N = 5$. For the Neumann problem, Figure 3(a) also shows that spurious eigenvalues appear using the real-part BEM ($N = 5$). It is emphasized that the imaginary-part BEM ($N = 5$) results in spurious eigenvalues for the Neumann problem as shown in Figure 4(a). This interesting case was not found in Kang's paper since they did not deal with the Neumann cases. As N becomes large, ill-posed behavior occurs and will be discussed analytically in the following.

(2) *Ill-conditioned behavior*: The Kang's approach is an ill-posed model since the determinant is found to be very small of order, 10^{-40}, 10^{-100}, 10^{-120}, using the direct search method as shown in Figures 4, 7 and 12 in reference [1] respectively. The inverse for the SM matrix may be difficult to computation. The range of k is set to be $2 < k < 9$, $4 < k < 11$ and $3 < k < 7$ in Figures 4, 7 and 12 of reference [1], respectively. According to the analytical study for a circular case, the ill-posed problem occurs seriously if the range of k is small since the condition number can be determined analytically by

$$C_n = \frac{\max(h_1, h_2, \ldots, h_N)}{\min(h_1, h_2, \ldots, h_N)}, \tag{55}$$

where h can be λ, μ, v or κ as shown in Eqs. (33)–(36). The ill-conditioned behavior becomes more serious when the number of elements, $2N$, is larger. The ill-posed problem was not significant in reference [1] since the maximum number of elements was only 24. Also, a smaller increment for k value will make the determinant deteriorate more seriously. For the Dirichlet problem, Figure 1(b) shows that the real-part BEM is not sensitive for the condition number while the ill-conditioned behavior occurs using the imaginary-part BEM ($N = 15$) in Figure 2(b). It is fortunate that the numerical instability was not found in Kang's paper since the maximum number of elements is 24 ($N = 12$) only. For the Neumann problem, similar results can be found in Figures 3(b) and 4(b). The local maximum disappears in Figures 2(b) and 4(b) since the numerical instability occurs due to ill-conditioned behavior for $N = 15$. To overcome the ill-posed problem for larger values of N, GSVD technique is one alternative [16].

(3) *Multiplicity*: To find the eigenvalues, the direct search method was employed in reference [1]. Therefore, the multiplicity cannot be identified. For the double roots, more effort is required to find the non-trivial vector of A_j. To solve this problem, the SVD



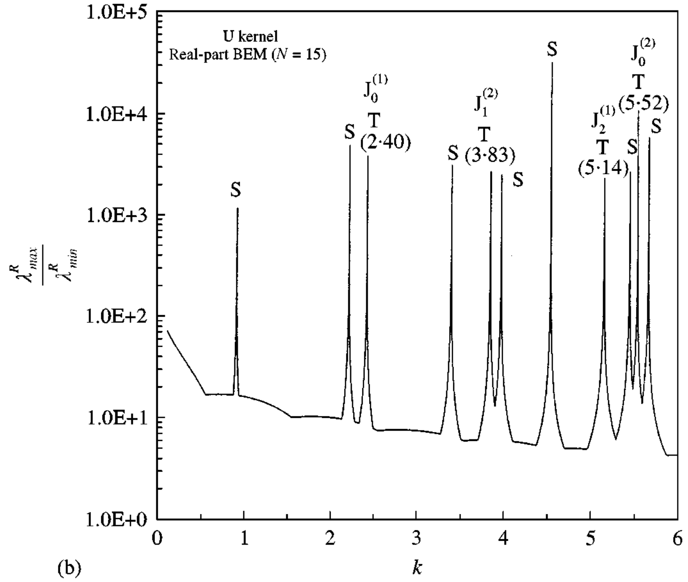


Figure 1. Condition number versus k using the real-part BEM for the Dirichlet problem. (a) $N = 5$, (b) $N = 15$: T—true eigenvalue; s—spurious eigenvalue.

technique can be employed to find the number of zeros for the singular values. The number of zeros is the multiplicity. The boundary modes can be extracted from the unitary matrix in SVD. To determine the true multiplicity, dual BEM is one alternative. More details can be found in references [7, 9, 15]. However, the spurious multiplicity occurs since spurious solution happens to be true for the Dirichlet case in reference [1].

(4) *Limitation—failure when applied to problems with a multiply connected domain*: Since the Kang's method has only the bases of $J_0(kr)$, we wonder whether the problem with a multiply connected domain can be solved successfully. For the problems with impedance boundary conditions and exterior problems with radiation conditions, the Kang's method

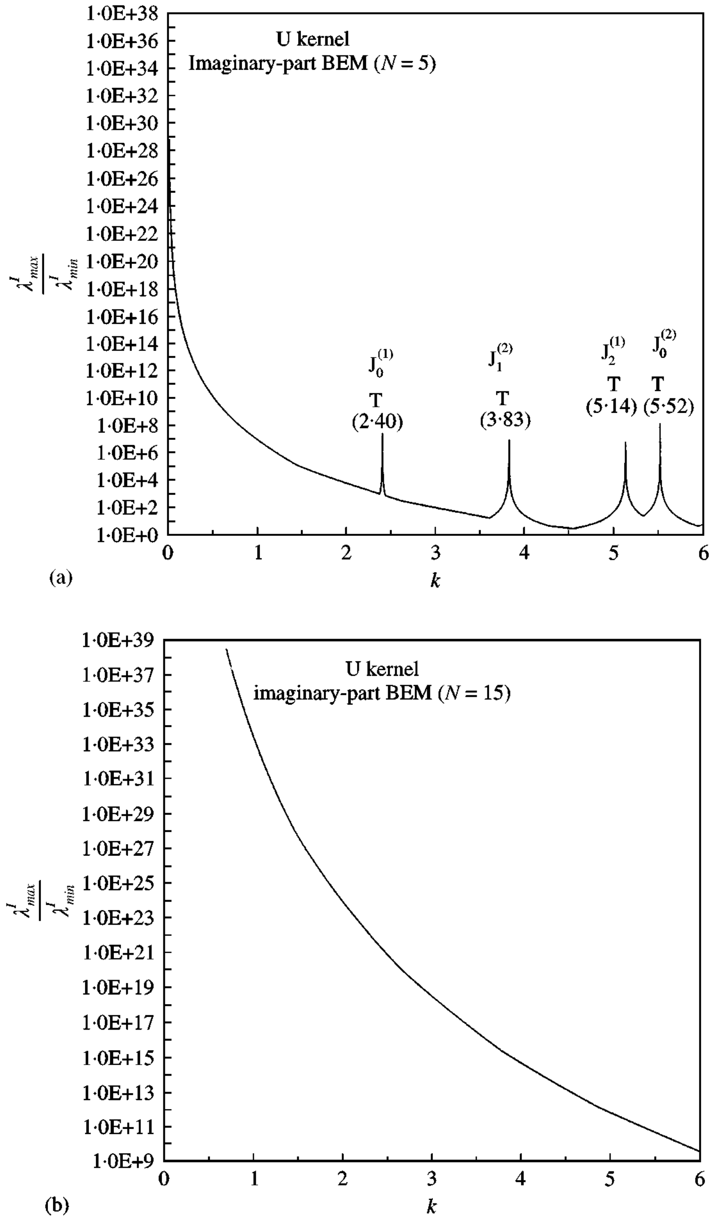


Figure 2. Condition number versus k using the imaginary-part BEM for the Dirichlet problem. (a) $N = 5$: T—true eigenvalue, (b) $N = 15$.

fails since no complex information is imbedded. We will prove the failure of the Kang's method when applied to solve an annular example [21] in Figure 5. The inner and outer radii are b and a respectively. For an annular domain with the Dirichlet boundary condition, $u_1 = 0$ and $u_2 = 0$, the discretizing equation is reduced to

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{56}$$

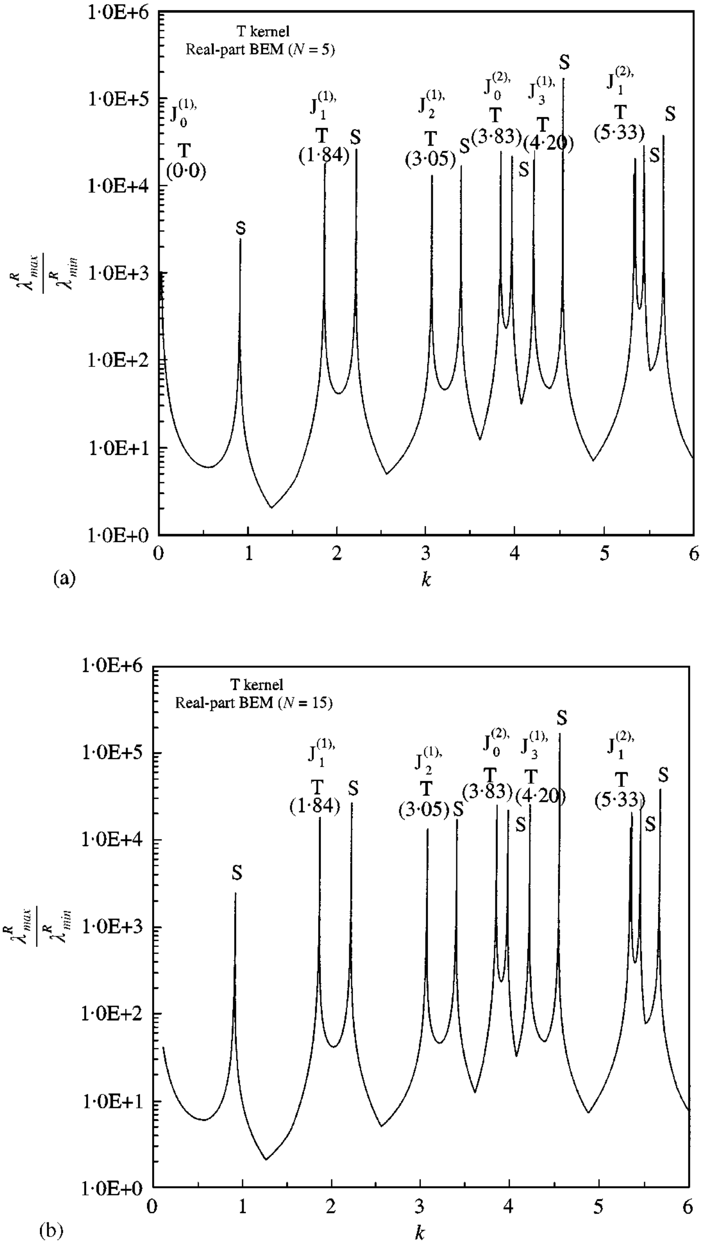
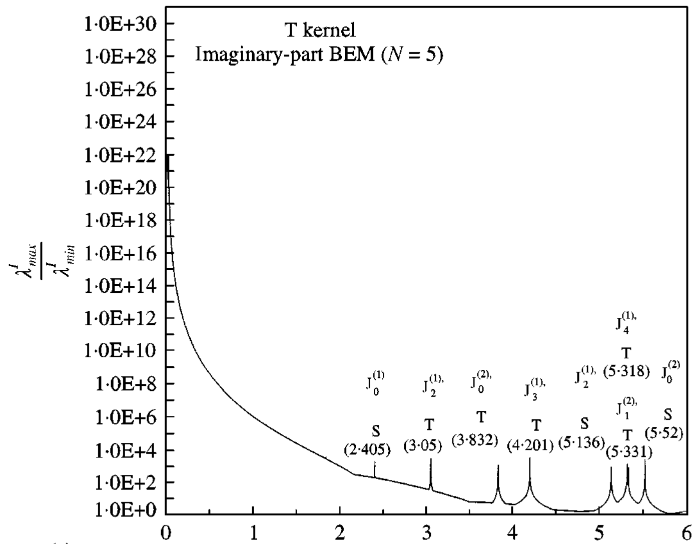


Figure 3. Condition number versus k using the real-part BEM for the Neumann problem. (a) $N = 5$, (b) $N = 15$: T—true eigenvalues; s—spurious eigenvalue.

where A, B, C and D are found to be circulants in a similar way as in equations (9)–(12). Similarly, we can decompose the four matrices into

$$[A] = [R][\bar{A}][R]^{-1}, \qquad [B] = [R][\bar{B}][R]^{-1}, \tag{57, 58}$$

$$[C] = [R][\bar{C}][R]^{-1}, \qquad [D] = [R][\bar{D}][R]^{-1}, \tag{59, 60}$$



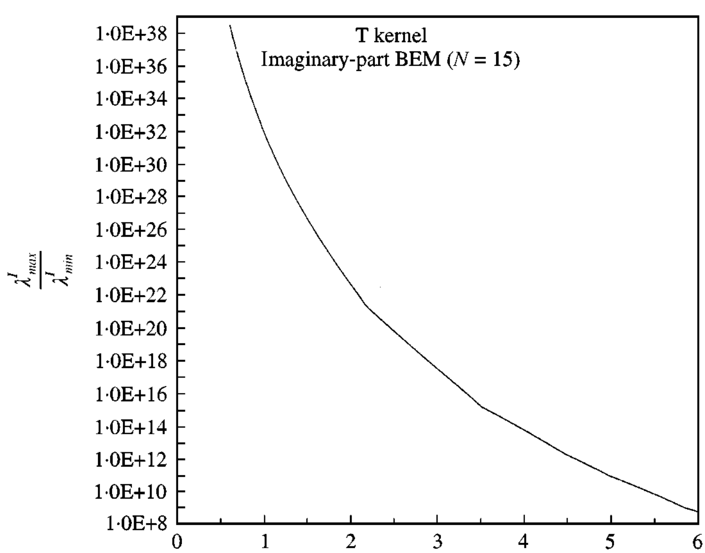


Figure 4. Condition number versus k using the imaginary-part BEM for the Neumann problem. (a) $N = 5$: T—true eigenvalue, s—spurious eigenvalue, (b) $N = 15$.

where

$$[R] = [\phi_0, \phi_1, \ldots, \phi_{2N-1}] \tag{61}$$

and $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are the diagonal matrices with the following elements:

$$\lambda_l^{[A]} = \frac{-S}{\rho \Delta\theta} \pi^2 a \mathrm{J}_l^2(ka), \qquad \lambda_l^{[B]} = \frac{-S}{\rho \Delta\theta} \pi^2 b \mathrm{J}_l(ka) \mathrm{J}_l(kb), \tag{62, 63}$$

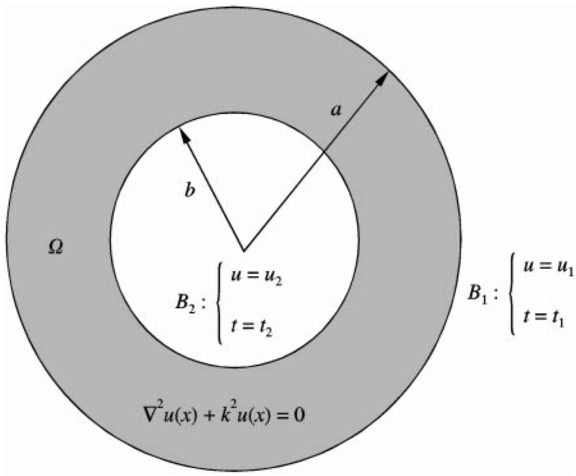


Figure 5. Helmholtz eigenproblem for an annular domain region.

$$\lambda_l^{[C]} = \frac{-S}{\rho \Delta\theta} \pi^2 a \mathrm{J}_l(ka)\mathrm{J}_l(kb), \qquad \lambda_l^{[D]} = \frac{-S}{\rho\Delta\theta}\pi^2 b \mathrm{J}_l^2(kb). \tag{64, 65}$$

Then the determinant of the matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

can be obtained:

$$det\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right| = det\begin{vmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{vmatrix} \det|[R]^{-1}[R]| \tag{66}$$

$$= det\begin{vmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{vmatrix}$$

$$= \prod_{l=0}^{2N-1} det\begin{vmatrix} \lambda_l^{[A]} & \lambda_l^{[B]} \\ \lambda_l^{[C]} & \lambda_l^{[D]} \end{vmatrix}$$

$$= \prod_{l=0}^{2N-1} \frac{S^2}{\rho^2(\Delta\theta)^2}(\pi^2)^2(ab)^2 \mathrm{J}_l(ka)J_1(kb)\begin{vmatrix} \mathrm{J}_l(ka) & \mathrm{J}_l(kb) \\ \mathrm{J}_l(ka) & \mathrm{J}_l(kb) \end{vmatrix}.$$

Then we can obtain the possible eigenequations,

$$\mathrm{J}_l(ka) = 0, \qquad \mathrm{J}_l(kb) = 0, \qquad \mathrm{J}_l(ka)\mathrm{J}_l(kb) - \mathrm{J}_l(ka)\mathrm{J}_l(kb) = 0. \tag{67–69}$$

The three eigenequations, equations (67)–(69), are spurious since the true eigenequation is [21]

$$\mathrm{J}_l(kb)\,Y_l(ka) - Y_l(kb)\mathrm{J}_l(ka) = 0. \tag{70}$$

Particularly, equation (69) is found to be trivial. Obviously, only the imaginary-part bases cannot solve the multiply connected problems. To deal with this problem, either the complex-valued dual BEM or the real-part dual BEM can be employed to solve the true eigensolutions. More details can be found in reference [22].

Concluding remarks: As mentioned by Kang *et al.* [1], the non-dimensional influence function method can be "expectably" applied to the multiply connected problems and cases with general boundary conditions. Based on the theories of degenerate kernels and circulants, we have proved that the Kang's method fails for the two cases. For the multiply connected domain problem of an annular region, three spurious eigenequations including one trivial case were obtained. For the Neumann problem of simply connected domain, the Kang's method was proved to have the problems of spurious eigensolutions. Also, the ill-posed problems and multiplicity for the true eigenvalues were addressed. In order to deal with the above difficulties, a series of works by NTOU-BEM Group can be found in greater details from the references.

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AUTHORS' REPLY

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The authors take a great interest in Dr Chen's comment in which various problems related to the application of our paper [1] have been addressed. The problems have been settled in our own way immediately after publishing the paper. Correspondingly, the concerned papers were submitted and will be soon published in well-known journal papers. Our review opinion on the comment is as follows. Dr Chen largely pointed out the four problems from our paper. (1) *Spurious eigensolutions*: it is correct that the spurious eigensolutions are produced when the method using the non-dimensional dynamic influence function has been extended to the Neumann problems. But in paper [1], the subject of analysis of interest was limited within membranes, for which the Neumann boundary condition is generally meaningless. In addition, how we settle this spurious problem in acoustic cavities with the Neumann boundary will be addressed in a paper to be published soon. (2) *Ill-conditioned behavior*: it is apparent that the NDIF method yields the ill-conditioned behavior when the boundary nodes are increased to obtain higher order modes. Note, however, that this problem is produced in the low-frequency range where lower order modes exist, not in the high-frequency range. The reason is that too excessive boundary nodes have been used to obtain lower order modes. Concretely speaking, in such a case when too many nodes are used, the ill-conditioned behavior is observed in only the low-frequency range and is out of the question because the converged eigenvalues for lower order modes are obtained when decreasing the number of nodes. (3) *Multiplicity*: The singular-value decomposition method has already been used in our past works (but the results were omitted in paper [1]). In this case the capability of search of eigenvalues was not good in the low-frequency range in comparison with the determinant searching method